PHYH-C IX: ELEMENTS OF MODERN PHYSICS

Solution for Assignment -I

05-March-2019

Answer all the questions

1. What is the value of Planck's constant. Write down the energy of a photon in terms of its wavelength and velocity. Prove that $\frac{h}{p}$ has dimension of length.

Marks: 6

Answer: $h = 6.626 \times 10^{-34}$ Joule-Second.

$E = \frac{hc}{\lambda}$, where h is Plank Constant, c is velocity of photon, λ wavelength of light.

Dimension of $h = J * s = kg * m * s^{-2} * m * s = ML^2T^{-1}$

dimension of $p = kg * m * s^{-1} = MLT^{-1}$

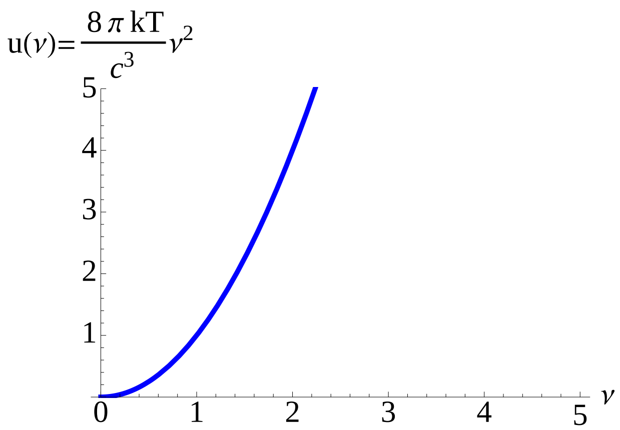
Thus $[\frac{h}{p}] = [L]$

2. Write down expression for blackbody spectrum derived by Lord Rayleigh and James Jeans. Plot their result of spectral energy density as a function of frequency. What is **Ultraviolet Catastrophe**?

Marks: 6

Rayleigh-Jeans result is

$$u_\nu d\nu = \frac{8\pi k_B T}{c^3}\nu^2 d\nu \tag{1}$$



As Rayleigh-Jeans law suggests, the energy density increases with increase in frequency. But as observed from experiments, the energy density decreases exponentially at large frequency or at small wavelength (thus the name Ultraviolet!). The failure of the Rayleigh-Jeans law at large frequency is known as Ultraviolet Catastrophe.

3. Write down Planck's Radiation formula and plot the energy density as a function of frequency for different value of temperature of the blackbody.

Marks: 6

All of you have done this correctly

4. Prove by simple mathematical approximations that Planck's radiation formula save the theoretical physics for both at small and high frequency and thus matching Rayleigh-Jeans result and no more UV catastrophe at high frequency.

Marks: 6

All of you have done this correctly

5. Suppose $[A, B] = AB - BA$, where A and B are mathematical operators. Take $A \equiv x$ and $B \equiv \hat{p} = -i\hbar\frac{\partial}{\partial x}$. Prove $[x, p]\psi(x) = i\hbar\psi(x)$

Marks: 6

$$[x, \hat{p}]\psi(x) = x(-i\hbar\frac{\partial}{\partial x})\psi(x) - (-i\hbar\frac{\partial}{\partial x})x\psi(x) \tag{2}$$

$$= -i\hbar x\frac{\partial \psi}{\partial x} + i\hbar\frac{\partial(x\psi)}{\partial x} = -i\hbar x\frac{\partial \psi}{\partial x} + i\hbar x\frac{\partial(\psi)}{\partial x} + i\hbar\psi = i\hbar\psi \tag{3}$$

thus proved.

6. A free particle is inside a infinitely rigid box with walls at $x = 0$ and $x = L$. There is no potential (thus no interaction!) inside the box and the particle is free to move inside. Take the mass of the particle m, find the discrete energies and frequencies and the normalised wave functions of the particle

Marks: 10

All of you have done this correctly

7. Redo the above problem, for the box with walls at $x = -\frac{L}{2}$ and $x = +\frac{L}{2}$. Clearly write down the differences in the results by making a table.

Marks: 10

Answer

Solution for the wave function and ω is given as

$$\psi(x) = A\sin(\omega x) + B\cos(\omega x); \qquad \omega^2 = \frac{2mE}{\hbar^2} \tag{4}$$

With the boundary conditions

$$\psi(x = -L/2) = 0; \qquad \psi(x = L/2) = 0 \tag{5}$$

Now we have the following equations

$$A\sin(\omega L/2) + B\cos(\omega L/2) = 0 \tag{6}$$
$$-A\sin(\omega L/2) + B\cos(\omega L/2) = 0 \tag{7}$$

Adding them we get,

$$B\cos(\omega L/2) = 0 \tag{8}$$

So either $B = 0$ or $\omega L/2 = (2n-1)\pi/2$ or $\omega = \frac{(2n-1)\pi}{L} = \omega_{odd}$. Here $n = 1, 2, 3\cdots$
Subtracting the equations we get

$$A\sin(\omega L/2) = 0 \tag{9}$$

from this we get either $A = 0$ or $\omega L/2 = n\pi$ or $\omega = \frac{2n\pi}{L} = \omega_{even}$, Here $n = 1, 2, 3\cdots$
From here we must understand the symmetry of the solution is more prominent in this boundary conditions. We have the solutions with

$$B = 0; A \neq 0; \omega = \omega_{even}$$

And the other solution with

$$A = 0; B \neq 0; \omega = \omega_{odd}$$

The two solutions are

$$\psi(x) = A\sin(\omega_{even} x) \tag{10}$$

And the other solution is

$$\psi(x) = B\cos(\omega_{odd} x) \tag{11}$$

Reminder, the ground state is $n = 1$, the odd solution, the first excited state is for $n = 2$. So

$$\psi_1 = B\cos(\frac{\pi x}{L}) \tag{12}$$
$$\psi_2 = A\sin(\frac{2\pi x}{L}) \tag{13}$$
$$\psi_3 = B\cos(\frac{3\pi x}{L}) \tag{14}$$
$$\psi_4 = A\sin(\frac{4\pi x}{L}) \tag{15}$$
$$\psi_5 = B\cos(\frac{5\pi x}{L}) \tag{16}$$

Keep in mind that the energy values are same as before, and also the nature of the wave function. These are not changing with shifting the boundary from $0 < x < L$ to $-L/2 < x < L/2$

I understand one can find the constants A and B from normalising the wavefunction from $-L/2 < x < L/2$. i.e.; do the integration

$$\int_{-\infty}^{\infty} |\psi(x)|^2 dx = 1 \tag{17}$$

Method-2, Just take your solution from problem 6 and make $x \to x + L/2$, you must get the solution for the wavefunctions.